The Interlace Polynomial and the Tutte-Martin Polynomial

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Abstract

This chapter introduces the interlace polynomial and the Tutte Martin polynomials, describes their main properties, and describes their relations with the Tutte polynomial and other polynomials.

1 Introduction

The interlace polynomial was discovered by Arratia, Bollobás, and Sorkin [5, 6] by studying DNA sequencing methods. Its definition can be traced from 4-regular graphs (or 2-in, 2-out digraphs), to circle graphs and finally to arbitrary graphs (multiple edges are not allowed). We take the same route. In Section 2 we consider the theory of circuit partitions in 4-regular graphs as initiated by the seminal paper of Kotzig [35], and consider the Martin polynomial [38] for 4-regular graphs (and 2-in, 2-out digraphs), which counts, for any k, the number a_k of circuit partitions of cardinality k. In Section 3 we associate a circle graph to any Eulerian circuit of a 4-regular graph, and we invoke a theorem by Cohn-Lempel-Traldi [24, 42] to recover a_k from its circle graph. This leads naturally to the definition of the interlace polynomial for arbitrary graphs.

We show that the interlace polynomial satisfies recursive relations involving the graph operations of local complementation and local edge complementation and we provide some evaluations of the interlace polynomial. It turns out that various polynomials are closely related to interlace polynomial. For example,

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we discuss the "global" interlace polynomial of Aigner and van der Holst [3]. Also, in Section 4, we show its close relationship with the Tutte polynomial for binary matroids.

Isotropic systems were introduced by Bouchet [10] to unify various properties of circuit partitions in 4-regular graphs and properties of pairs of duals of binary matroids. The Tutte-Martin polynomials for isotropic systems [13] were defined about a decade before the introduction of the interlace polynomial, and once the preliminary conference paper of the interlace polynomial appeared, various authors quickly noticed that the interlace polynomial can be seen as a special case of the restricted Tutte-Martin polynomial [3, 15]. In Section 5 we discuss the Tutte-Martin polynomials and their relationship with interlace polynomials.

Sections 6 and 7 discuss other generalizations of the interlace polynomial, and open problems are given in Section 8. Note that we do not respect chronological order in this chapter as we define the interlace polynomial before the Tutte-Martin polynomial — the reason is that the former can be defined using only elementary notions, while the latter relies on the more involved notion of isotropic system.

2 4-Regular Graphs and the Martin Polynomial

In this section we consider 4-regular graphs, as well as the directed 2-in 2-out variant. The polynomial we describe here was first defined recursively by Martin [38], and Las Vergnas [36] later obtained the explicit formulation we use here. Interestingly, this happened in the same order for the interlace polynomial, see Section 3.3. A *circuit partition* of a graph (or digraph) G is a partition of the edges of G into circuits.

The Martin polynomial of a (di)graph counts the number of circuits over all circuit partitions of a graph. When tracing circuits in 2-in 2-out graphs, after entering a vertex we can leave that vertex in two ways following either one of the outgoing edges. In 4-regular graphs there are generally four edges at each vertex, and we can choose three directions to continue a circuit. We want to keep this property when a vertex is incident to a loop. Thus we have the premise that an undirected loop may be entered from two sides. This leads to a slightly more precise concept to cover the concept of circuit partition.

When discussing 2-in 2-out graphs and 4-regular graphs we allow both loops and multiple edges, i.e., we actually consider multigraphs.

Definition 2.1. A *transition* at a vertex v is a partition in pairs of the half-edges incident to v. For a digraph we require an orientation-consistent pairing. A *transition system* (sometimes called a *graph state*) of G is the assignment of a transition to each vertex of G.

Let $\mathcal{T}(G)$ be the set of transition systems of (di)graph G. For a transition system $T \in \mathcal{T}(G)$, let $k(T)$ be the number of circuits in the circuit partition of G induced by T. Moreover, let $c(G)$ be the number of connected components

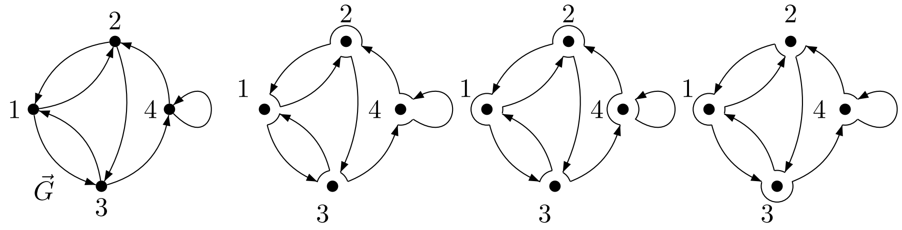

Figure 1: Digraph $\vec{G}$ and three of sixteen possible transition systems.

of G. Finally, an *Eulerian system* of G is a set containing exactly one Eulerian circuit for each connected component of G.

2.1 Directed case: 2-in 2-out graphs

Definition 2.2. Let $\vec{G}$ be a 2-in 2-out digraph. The *Martin polynomial* of $\vec{G}$ is defined as $m(\vec{G}; y) = \sum_{T \in \mathcal{T}(\vec{G})} (y-1)^{k(T)-c(\vec{G})}$.

We remark that Definition 2.2 is commonly defined only for *connected* 2-in 2-out digraphs, i.e., the case $c(\vec{G}) = 1$. However, its generalization to arbitrary 2-in 2-out digraphs is straightforward.

Example 2.3. In Figure 1 we consider a 2-in 2-out graph $\vec{G}$ together with a representation of three of its 16 transition systems. The induced circuit partitions consist of 2, 2, and 1 circuits, respectively. Tallying the circuit count for all partitions we obtain $m(\vec{G}; y) = 3(y-1)^0 + 7(y-1)^1 + 5(y-1)^2 + 1(y-1)^3 = y^3 + 2y^2$; e.g., there are three different Eulerian circuits for $\vec{G}$.

We start with a number of evaluations of the Martin polynomial. For any 2-in 2-out digraph $\vec{G}$, there is a unique circuit partition P in the corresponding (undirected) graph G such that for each circuit C of P, the edges of C alternate with respect to its direction given by $\vec{G}$. The circuits of P are called *anticircuits* and the number of anticircuits of $\vec{G}$ is denoted by $a(\vec{G})$.

Theorem 2.4 ([38, 36, 37])**.** *Let $\vec{G}$ be a 2-in 2-out digraph and $n = |V(\vec{G})|$.*

- $m(\vec{G}; -1) = (-1)^n(-2)^{a(\vec{G})-1}$.
- $m(\vec{G}; 0) = 0$, *when* $n > 0$.
- $m(\vec{G}; 1)$ *is the number of Eulerian systems of* $\vec{G}$,
- $m(\vec{G}; 2) = 2^n$,
- $m(\vec{G}; 3) = k\,|m(\vec{G}; -1)|$ *for an odd integer* k.

The Martin polynomials may be computed using a *recursive* approach. The *vertex reduction* at a transition t at a vertex v merges each of the edge pairs from

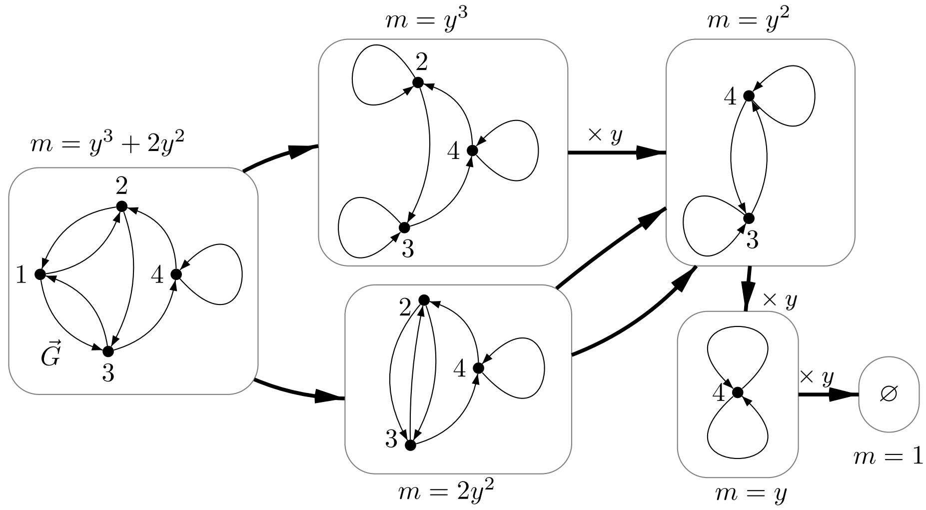

Figure 2: Computation of the Martin polynomial by vertex reduction, see Example 2.6. With each graph $\vec{F}$ we give the polynomial $m = m(\vec{F}, y)$.

t into one edge, and deletes v. Vertex reduction is not defined for transitions that would obtain an edge not incident to any vertex (such as transitions that pair both half-edges of a loop).

Recall that a *cut vertex* is a vertex v that by cutting/splitting v obtains a graph with a larger number of connected components. In this way a looped vertex is considered a cut vertex, the loop being one side of the cut.

Theorem 2.5 ([38])**.** *Let $\vec{G}$ be a 2-in 2-out digraph.*

- *If $v(\vec{G}) = 0$, then $m(\vec{G}; y) = 1$.*
- *If v is a cut vertex, then $m(\vec{G}; y) = y\, m(\vec{G}'; y)$, where $\vec{G}'$ is the digraph obtained by applying the vertex reduction at the transition at v that does not increase the number of connected components.*
- *If v is a vertex without loops, then $m(G; y) = m(\vec{G}'_v; y) + m(\vec{G}''_v; y)$, where $\vec{G}'_v$ and $\vec{G}''_v$ are the two graphs obtained by applying the two vertex reductions at v.*

For a cut vertex without loops, both of the last two cases of Theorem 2.5 apply, and give (of course) the same result. In this case one of the transitions splits the graph and always has one circuit more than when the other transition is taken. Hence $m(\vec{G}'_v; y) = (y-1)m(\vec{G}''_v; y)$.

Example 2.6. Let $\vec{G}$ be the (leftmost) digraph from Figure 1. We compute $m(\vec{G}; y)$ recursively, using Theorem 2.5, see Figure 2. We obtain $m(\vec{G}; y) = (y+2)y^2$. By Theorem 2.4, $\vec{G}$ has $m(\vec{G}; 1) = 3$ Eulerian circuits. Finally, we verify that $m(\vec{G}; 2) = 2^4$, while $m(\vec{G}; -1) = 1 = (-1)^4(-2)^{1-1}$ matching the only anticircuit of $\vec{G}$.

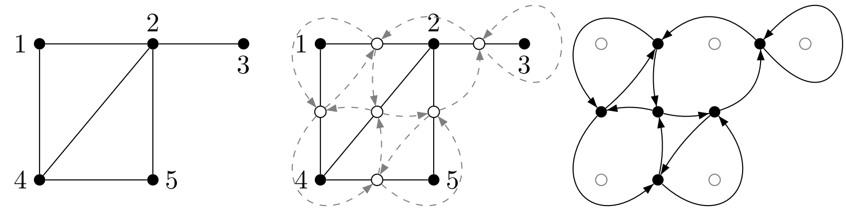

Figure 3: Constructing the directed medial graph $\vec{G}_m$. New vertices correspond to edges in the original graph, new edges are counterclockwise around the original vertices. The final result is given right, with the "shadow" of the original vertices for reference.

Given a plane graph G, its directed *medial graph* $\vec{G}_m$ is constructed as follows. For each edge of G, create a vertex for $\vec{G}_m$. For each vertex v of G, create a directed edge between any two vertices of $\vec{G}_m$ where the corresponding edges in G are consecutive in the anti-clockwise ordering of the edges incident to v, cf. Figure 3. The directed medial graph is 2-in 2-out.

The following result connects the Martin polynomial with the Tutte polynomial.

Theorem 2.7 ([38]). *Given a plane graph G, with its directed medial graph $\vec{G}_m$, we have $m(\vec{G}_m; y) = T(G; y, y)$.*

It is instructive to have another look at Figure 3 to verify that deletion and contraction of an edge correspond to the two vertex reductions at the vertex corresponding to the edge in the medial graph. In fact the digraphs of Figure 2 can be seen as medial graphs of plane graphs. It can be observed that both loops and bridges in G correspond to cut vertices in the medial graph $\vec{G}_m$, for which Theorem 2.5 holds.

We invite the reader to have a quick peek at Theorem 4.1, where a more general version of this Tutte-Martin connection is given.

2.2 Undirected case: 4-regular graphs

Definition 2.8. Let G be a 4-regular graph. The *Martin polynomial* of G is defined as $M(G; y) = \sum_{T \in \mathcal{T}(G)} (y-2)^{k(T)-c(G)}$.

The difference in definition between 2-in 2-out graphs and 4-regular graphs is caused by the fact that the definitions of the Martin polynomials were originally given in [38] in a recursive fashion, with the explicit closed formula appearing only later in [36].

Theorem 2.9 ([38]). *Let G be a 4-regular graph.*

- *If $v(G) = 0$, then $M(G; y) = 1$.*

- *If v is a cut vertex, then $M(G;y) = y\,M(G';y)$, where G' is the graph obtained by applying a vertex reduction at a transition at v that does not increase the number of connected components.*
- *If v is a vertex without loops, then $M(G;y) = M(G'_v;y) + M(G''_v;y) + M(G'''_v;y)$, where G'_v, G''_v, and G'''_v are the graphs obtained by applying the three vertex reductions at v.*

For the undirected case we again have the formula $M(G;x) = y\,M(G';x)$, even though we now add three polynomials (one of which results from a transition that will disconnect the sides of the graph at the cut). This is a consequence of choosing $y-2$ instead of $y-1$ in Definition 2.8.

The distinction between two and three continuations will be a recurring theme in this chapter, even in later, more abstract, situations where we no longer recognize "directions".

Example 2.10. We have $M(G;y) = y^2(y+6)$, where G is the undirected version of $\vec{G}$, from Example 2.6, Figure 2.

Theorem 2.11 ([38])**.** *Let G be a 4-regular graph and $n = |V(G)|$.*

- *$M(G;0) = 0$, when $n > 0$.*
- *$M(G;2)$ is the number of Eulerian systems of G,*
- *$M(G;3) = 3^n$.*

2.3 Weighted polynomials

A *transition weight function* W is a mapping that assigns a weight $W(t)$ to each transition t at each vertex. The weight $W(T)$ of a transition system T is then the product $\prod_{t\in T} W(t)$ of the weights of all transitions t in T.

Definition 2.12 ([34])**.** Given a transition weight function W, the (weighted) *transition polynomial* of 4-regular graph G is defined as

$$M(G,W;y) = \sum_{T\in\mathcal{T}(G)} W(T)\,y^{k(T)-c(G)}.$$

Jaeger [34] observes that the transition polynomial for 4-regular graphs captures both Martin polynomials and the Penrose polynomial. Setting all transition weights to 1 we have $M(G,W;y) = yM(G;y+2)$. For a 2-in 2-out digraph $\vec{G}$, we can consider its underlying (undirected) graph G and assign weight 0 to transitions that join edges against the original orientation and 1 otherwise. Then $M(G,W;y) = m(\vec{G};y+1)$. The Penrose polynomial [39] (see also [1]) is defined for a plane graph, and is based on the circuits of its medial graph. Setting weight -1 for transitions connecting opposite half-edges, weight $+1$ for transitions connecting the same side of the original edges of G, and weight 0 otherwise, we obtain the Penrose polynomial as a special case of $M(G,W;y)$ [34].

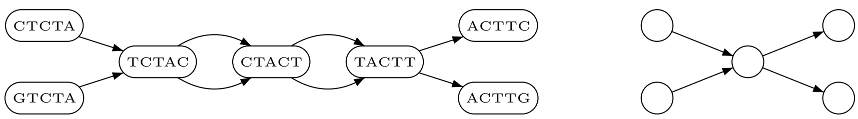

Figure 4: De Bruijn graph, section for a string $\cdots$GTCTACTTG$\cdots$CTCTACTTC$\cdots$ with $\ell = 6$ (left) with its contraction (right).

Of course, the transition polynomial may be defined for arbitrary Eulerian (di)graphs G. Setting all transition weights of the transition polynomial to 1 and multiplying by $y^{c(G)}$, we obtain the *circuit partition polynomial* $r(G; y) = \sum_{T \in \mathcal{T}(G)} y^{k(T)}$. The circuit partition polynomial has been studied in [9, 28, 29] for arbitrary Eulerian (di)graphs G.

2.4 De Bruijn Graphs for DNA Sequencing

In sequencing by hybridization one tries to determine a long strand S of DNA from its *ℓ-spectrum*, which is the multiset of all substrings of S of length ℓ, for some constant ℓ. The substrings can be found by testing whether probes for all sequences of length ℓ match the DNA (modern techniques are able to read DNA segments of small length directly). The algorithm is based on constructing a *de Bruijn graph* for S [25]. The edges of this graph are exactly the length ℓ strings of the spectrum of S, while the vertices are the length $\ell - 1$ strings: edge axb runs from ax to xb (where a and b are the first and last symbols). When S contains one of more "interlaced repeats" the reconstruction is no longer unique: the spectra of $S_1 = z_1\, x\, z_2\, y\, z_3\, x\, z_4\, y\, z_5$ and $S_2 = z_1\, x\, z_4\, y\, z_3\, x\, z_2\, y\, z_5$ are equal (assuming x and y are segments of length at least ℓ). Arratia et al. [4] study the number of possible reconstructions of spectra, and observe that it is useful to work with a 2-in 2-out graph constructed from the de Bruijn graph (where they assume no string has multiplicity larger than three). Every sequence of vertices with parallel edges is contracted into a single vertex, see Figure 4, and every sequence of vertices with single out-edges is contracted into a single edge. For convenience, the initial and final vertices are identified. Circuits in this "macroscopic graph" correspond to reconstructions of the spectrum.

2.5 Gene Gymnastics in Ciliates

The ancient group of ciliates consists of unicellular organisms with a remarkable property: their DNA is stored in two types of nuclei. The germline micronucleus (MIC) contains a scrambled copy of the genes in the macronucleus (MAC) which is used for transcription. During conjugation the MIC is transformed into MAC in a process called *gene assembly*. To give an example (taken from Prescott [40]), the MIC version of the Actin I gene of Sterkiella nova can be represented as the string $I_0\, M_3\, I_1\, M_4\, I_2\, M_6\, I_3\, M_5\, I_4\, M_7\, I_5\, M_9\;\; I_6\, \overline{M}_2\, I_7\, M_1\, I_8\, M_8\, I_9$, where the

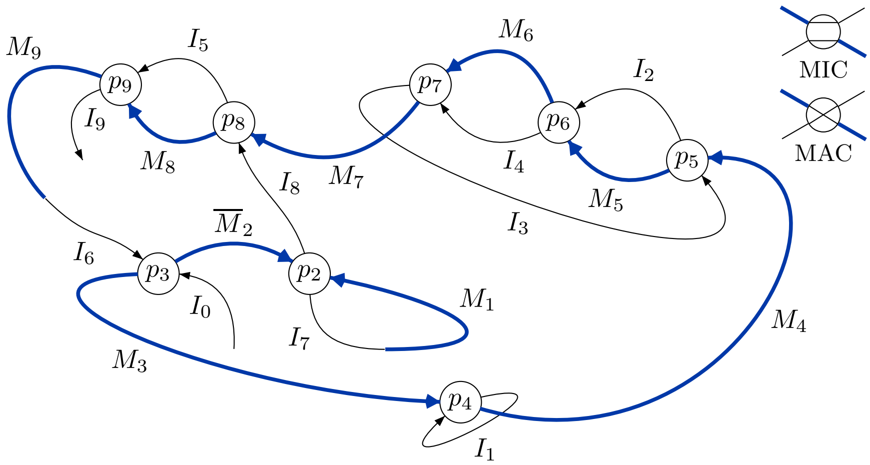

Figure 5: Actin I gene of Sterkiella nova. Schematic diagram, based on [40].

MAC version reads $M_1M_2\dots M_8M_9$. Here the M_i are so-called macronuclear destined sequences (MDSs for short) which are kept (but reordered) while the internal eliminated sequences E_i (IESs for short) are excised and digested during gene assembly.

The two versions of the gene can be naturally modeled as a 2-in 2-out graph, where one set of transitions defines a single circuit forming the MIC, while another set traces the MAC (with flanking IES's) and excised circular molecules, see Figure 5. Note that in the MAC some edges may be followed against their orientation which is the case if an MDS is inverted in the MIC representation, as M_2 in Actin I above (indicated by the bar).

Burns et al. [23] propose the *assembly polynomial* to capture the intermediate products in this gene rearrangement process.

3 Circle Graphs and the Interlace Polynomial

3.1 Preliminaries

Let us denote the rank and nullity of a matrix A by $r(A)$ and $n(A)$, respectively. Moreover, for a $V \times W$-matrix A indexed by finite sets V and W, we denote for $X \subseteq V$ and $Y \subseteq W$ the $X \times Y$-submatrix of A by $A[X,Y]$. Also, we denote $A[X,X]$ by $A[X]$.

With the exception of 2-in 2-out and 4-regular graphs, in this chapter we do not allow multiple edges for graphs, but we do allow loops.

For a graph G we use $A(G)$ to denote its adjacency matrix, which is viewed as a $V(G) \times V(G)$ matrix over $GF(2)$. For $Y \subseteq V(G)$, we write $G + Y$ for the graph that results from G after "toggling" loops at the vertices from Y. Thus, it has adjacency matrix $A(G) + I_Y$, where I_Y is the diagonal $V(G) \times V(G)$-matrix

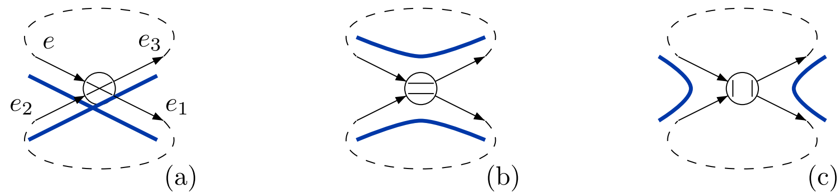

Figure 6: Three ways to connect pairs of edges in a 4-regular graph relative to an (oriented) Eulerian circuit: **(a)** following the circuit, **(b)** in an orientation-consistent way, and **(c)** in an orientation-inconsistent way.

over $GF(2)$ where for $i \in V$, the (i,i) entry is 1 if and only if $i \in Y$. In case $Y = \{v\}$ is a singleton, we also write $G + v$ for $G + Y$. For $X \subseteq V(G)$, we write $G[X]$ to denote the graph with adjacency matrix $A(G)[X]$.

3.2 Circle graphs

A *circle graph* is the intersection graph of chords in a circle: represent each chord by a vertex, and vertices are adjacent when their chords intersect. Given a Eulerian circuit C, we imagine the vertices of C placed along a circle, and connect the two occurrences by a chord. The corresponding circle graph is called the *interlace graph* $I(C)$ of C: two distinct vertices p, q are adjacent if they are *interlaced* in C, i.e., vertices occurring in the order $\cdots p \cdots q \cdots p \cdots q \cdots$ on C. For an Eulerian system C, its *interlace graph* $I(C)$ is the graph having the interlace graph for each Eulerian circuit of C as a connected component of $I(C)$. The connections between traversals of self-crossing plane curves and their "interlacement" properties goes back to the work of Gauss, cf. [27].

Relative to a fixed Eulerian circuit C, the transitions at a vertex can be unambiguously described. Fix an orientation of C. Then each transition either *follows* C (it equals the one chosen by C), or it differs, and is either *orientation consistent* or *inconsistent* (relative to the one chosen by C), cf. Figure 6. Hence a transition system can be specified by giving C and a partition of $V(G)$ into three subsets.

It turns out that the number of circuits in a circuit partition can be expressed as a nullity value related to the interlace graph.

Theorem 3.1 ([24, 42])**.** *Let G be a 4-regular graph with Eulerian system C. Let P be a circuit partition of $E(G)$, where D_1, D_2, D_3 are the sets of vertices that follow C, are orientation consistent, or are orientation inconsistent (respectively). Then $|P| - c(G) = n((I(C) + D_3) \setminus D_1)$.*

We remark that Theorem 3.1 is in fact as special case of a more general result which says that the dual of the circuit matroid of a graph called the *touch graph* (we do not recall this graph here) is equal to the column matroid of the matrix obtained from $I(C) + D_3$ by replacing all columns of D_1 by unit

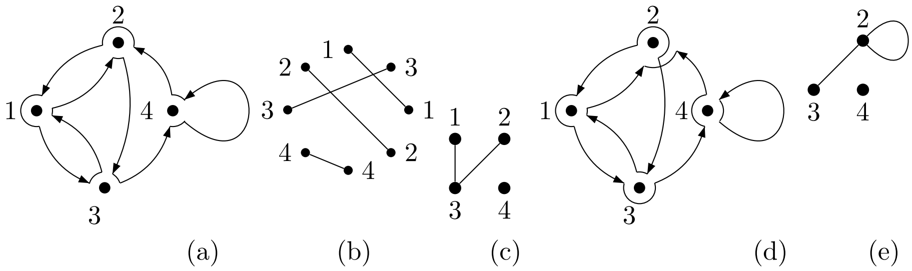

Figure 7: **(a)** An Eulerian circuit with **(b)** its chord diagram, and **(c)** its interlace graph **(d)** Circuit partition and **(e)** application of Theorem 3.1.

columns [33, 44]. Theorem 3.1 only says that the nullities of both matroids are equal.

Example 3.2. Let $C = 2\,1\,3\,1\,2\,3\,4\,4$ be the sequence of vertices visited along an Eulerian circuit. In Figure 7 we see the circuit (with edge orientations added for reference) and its interlace graph $I(C)$.

Now consider the circuit partition P which is described by $D_1 = \{1\}$, $D_2 = \{3, 4\}$, and $D_3 = \{2\}$ relative to C. It is depicted in Figure 7 together with the graph with adjacency matrix $I(C) + D_3 \setminus D_1$, cf. Theorem 3.1. Its nullity equals 1, hence P contains two circuits. Note that part of the largest circuit runs against the original orientation of C, due to the orientation-inconsistent transition at vertex 2.

3.3 Interlace polynomial

We are now ready to define the interlace polynomial.

Definition 3.3. Let G be a graph. Then the (single-variable) *interlace polynomial* of G is

$$q(G; y) = \sum_{X \subseteq V(G)} (y-1)^{n(A(G)[X])}.$$

The polynomial is sometimes known as the "vertex-nullity interlace polynomial" to distinguish it from its two-variable cousin, see Section 3.7, and to stress the fact that the summation is over sets of vertices of the graph (rather than edges, like for the Tutte polynomial).

Because of Theorem 3.1, we have a direct connection between the interlace polynomial and the Martin polynomial.

Theorem 3.4. *Let $\vec{G}$ be a 2-in 2-out graph, and let C be an Eulerian system for $\vec{G}$. Then $m(\vec{G}; y) = q(I(C); y)$.*

Example 3.5. Consider the graph G as depicted in Figure 8. Then $q(G; y) = (y-1)^2 + 6(y-1) + 9(y-1)^0 = y^2 + 4y + 4$. As an example, subset $\{a, d\}$ induces the 2×2 zero-matrix, which has nullity 2 and contribution $(y-1)^2$.

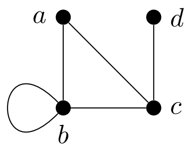

Figure 8: A graph.

We mention that explicit formulas have been obtained for the interlace polynomials of empty graphs, complete graphs, stars, complete bipartite graphs, and cycles [6].

Remark 3.6. We recall in Section 5 that the interlace polynomial can be seen as a special case of the more involved Tutte-Martin polynomial for isotropic systems. Many results regarding the interlace polynomial can be found in the context of the Tutte-Martin polynomial in [13], published almost a decade before the introduction of the interlace polynomial. However, since the notions of interlace polynomial and Tutte-Martin polynomial are sufficiently different, we here attribute results of the interlace polynomial to both [13] and the papers who proved the results in the context of interlace polynomials.

3.4 Recursive relations

Let v be a vertex of a graph $G = (V, E)$. The *neighborhood* of v in G, denoted by $N_G(v)$, is the set $\{w \in V \mid \{v, w\} \in E, w \neq v\}$. The *complement* of G is the graph G' obtained by complementing the edge relation, i.e., for every $e = \{v, w\}$ with $v, w \in V$ ($v = w$ is allowed), e is an edge of G if and only if e is not an edge of G'.

Let G be a graph and u a looped vertex of G, then the *local complement* of G at u, denoted by $G * u$, is the graph obtained from G by complementing the subgraph induced by the neighborhood of u.

Remark 3.7. Of course, we could just as easily define local complement for arbitrary vertices, but, as we will see, it turns out that it is convenient to restrict applicability of this operation to looped vertices. Similarly, we define the operation of edge local complement below only for edges having unlooped vertices.

The *closed neighborhood* of a vertex v in G, denoted by $\bar{N}_G(v)$, is the set $N_G(v) \cup \{v\}$. Let $e = \{v, w\} \in E(G)$ be an edge with v and w unlooped vertices, and consider the partition of $\bar{N}_G(v) \cup \bar{N}_G(w)$ into the sets $V_1 = \bar{N}_G(v) \setminus \bar{N}_G(w)$, $V_2 = \bar{N}_G(w) \setminus \bar{N}_G(v)$, and $V_3 = \bar{N}_G(v) \cap \bar{N}_G(w)$. Then the *edge local complement* of G at e, denoted by $G * e$, is the graph obtained from G by "complementing" the edges between distinct V_i's. Thus, for every $e' = \{x, y\}$ with $x \in V_i$, $y \in V_j$, and $i \neq j$, we have that e' is an edge of G if and only if e' is not an edge of $G * e$. The operation of edge local complementation is illustrated in Figure 9.

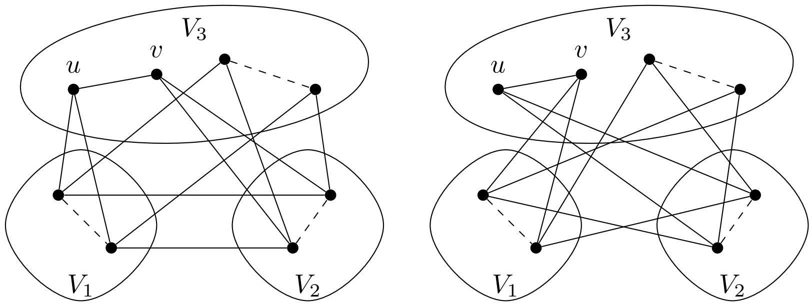

Figure 9: Edge local complementation on an edge $\{u, v\}$ in a graph. Note that u and v are adjacent to all vertices in V_3 — these edges are omitted in the diagram. The operation does not affect edges adjacent to vertices outside the sets V_1, V_2, V_3, nor does it change any of the loops.

The next theorem shows that the interlace polynomial satisfies recursive relations that *characterizes* the interlace polynomial. In fact, the original definition of [5] was given in this way.[1]

Theorem 3.8 ([6, 13])**.** *Let G be a graph.*

- *If $V(G) = \emptyset$, then $q(G; y) = 1$.*
- *If v is an isolated (unlooped) vertex of G, then $q(G; y) = y\, q(G \setminus v; y)$.*
- *If v is a looped vertex of G, then $q(G; y) = q(G \setminus v; y) + q((G * v) \setminus v; y)$.*
- *If $e = \{v, w\} \in E(G)$ with v and w unlooped vertices, then $q(G; y) = q(G \setminus v; y) + q((G * e) \setminus v; y)$.*

We remark that Theorem 3.8 essentially generalizes Theorem 2.5, where the latter corresponds to the case where $G = I(C)$ is a circle graph. For example, v is an isolated (unlooped) vertex of $I(C)$ if and only if v is a cut vertex in the underlying 2-in 2-out digraph. However, in Theorem 3.8 the different cases are disjoint (i.e., for each vertex, exactly one of the latter three conditions hold).

Example 3.9. In Figure 10 we illustrate the recursive computation of the interlace polynomial for the graph G from Figure 8, cf. Example 3.5. We obtain (again) $q(G; y) = y^2 + 4y + 1$. For simplicity, Figure 10 does not include the computations on graphs with only isolated unlooped vertices.

3.5 Other properties

An important property of the interlace polynomial is that it is invariant under both local complementation and edge local complementation. These are two

[1] To simplify various results, the definition of edge local complement presented here is slightly different from the definition in [6] (the only difference is that identities of the vertices of the edge e are swapped).

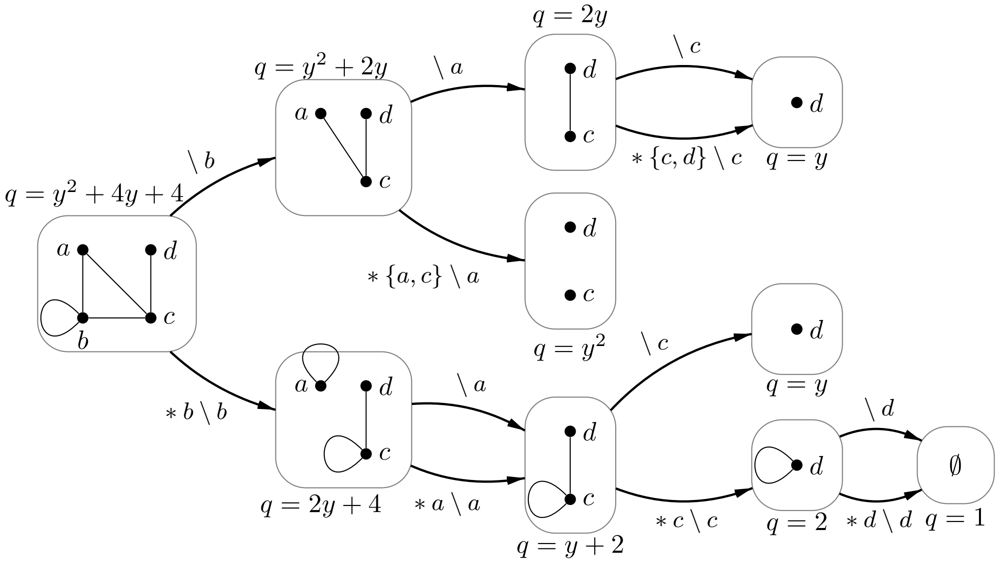

Figure 10: Recursive computation of the interlace polynomial $q(G; y)$. With each graph F we give the polynomial $q = q(F; y)$.

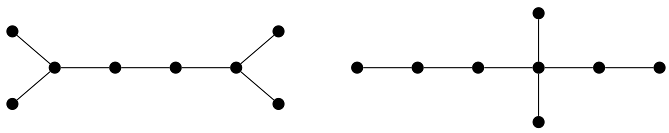

Figure 11: Two trees with the same interlace polynomial.

special cases of the more general invariance under principal pivot transform, see Theorem 6.2.

Theorem 3.10 ([7, 6])**.** *Let G be a graph.*

- *If v is a looped vertex of G, then $q(G; y) = q(G * v; y)$.*
- *If $e = \{v, w\} \in E(G)$ with v and w unlooped vertices, then $q(G; y) = q(G * e; y)$.*

Theorem 3.10 shows that graphs cannot be characterized by their interlace polynomials. In fact, the trees of Figure 11 have the same interlace polynomial, but are not in the same orbit under edge local complementation [3].

We now consider a number of evaluations of the interlace polynomial. A *perfect matching* of a graph G is a set of edges P (loops are allowed) of G such that every vertex of G is incident to exactly one edge of P. The next theorem is shown by Aigner and van der Holst [3]. They only consider simple graphs, but it is easy to see that the evaluations of $q(G;1)$ and $q(G;2)$ carry over trivially to graphs (with loops) and, using [13], it is observed in [22] that the evaluations of $q(G;-1)$ and $q(G;3)$ carry over to graphs as well. This is not true for $q(G;0)$.

Theorem 3.11 ([3, 13])**.** *Let G be a graph and $n = |V(G)|$.*

- $q(G;-1) = (-1)^n(-2)^{n(A(G)+I)}$, *where I is the $V(G) \times V(G)$ identity matrix,*
- $q(G;0) = 0$ *if $n > 0$ and G has no loops,*
- $q(G;1)$ *is equal to the number of induced subgraphs of G with an odd number of perfect matchings (including the empty graph),*
- $q(G;2) = 2^n$,
- $q(G;3) = k\,|q(G;-1)|$ *for some odd integer k.*

Evaluations and invariance properties of the interlace polynomial have also been investigated for particular subclasses of graphs. In particular, interlace polynomials for distance hereditary graphs have been investigated in [30].

3.6 The global interlace polynomial

Let us consider the following (unnamed) polynomial defined in [3]:

Definition 3.12. Let G be a graph. Then the *global interlace polynomial* of G is

$$Q(G;y) = \sum_{X \subseteq V(G)} \sum_{Y \subseteq X} (y-2)^{n((A(G+Y))[X])}.$$

The definition of the global interlace polynomial is motivated by Theorem 3.1. Whereas the interlace polynomial $q(G;y)$ for an interlace graph $G = I(C)$ only considers transitions that either follow C or are orientation consistent, this global variant allows all three possibilities. In this way, $q(G;y)$ generalizes the Martin polynomial for 2-in 2-out digraphs, Theorem 3.4, whereas $Q(G;y)$ generalizes the Martin polynomial for 4-regular graphs.

Theorem 3.13. *Let G be a 4-regular graph, and let C be an Eulerian system for G. Then $M(G;y) = Q(I(C);y)$.*

The next result presents invariant results for the global interlace polynomial.

Theorem 3.14 ([3, 13])**.** *Let G be a graph.*

- *If v is a vertex of G, then $Q(G;y) = Q(G+v;y)$.*
- *If v is a looped vertex of G, then $Q(G;y) = Q(G*v;y)$.*
- *If $e = \{v,w\} \in E(G)$ with v and w unlooped vertices, then $Q(G;y) = Q(G*e;y)$.*

The next theorem shows that there are recursive relations that *characterize* $Q(G; y)$ (along with the equality $Q(G; y) = Q(G + v; y)$ for vertices v of G).

Theorem 3.15 ([3, 13]). *Let G be a graph.*

- *If $V(G) = \emptyset$, then $Q(G; y) = 1$.*
- *If v is an isolated vertex of G, then $Q(G; y) = y\, Q(G \setminus v; y)$.*
- *If $e = \{v, w\} \in E(G)$ with v and w unlooped vertices, then $Q(G; y) = Q(G \setminus v; y) + Q((G * e) \setminus v; y) + Q(((G + v) * v) \setminus v; y)$.*

Since $Q(G; y)$ is invariant under adding/removing loops, we assume in the next result without loss of generality that G is simple (i.e., does not have any loops).

Theorem 3.16 ([3, 13]). *Let G be a simple graph and $n = |V(G)|$.*

- *$Q(G; 0) = 0$ if $n > 0$,*
- *$Q(G; 2)$ is the number of graphs G' (including the empty graph) such that (1) removing all loops from G' obtains an induced subgraph of G and (2) G' has an odd number of perfect matchings,*
- *$Q(G; 3) = 3^n$,*
- *$Q(G; 4) = 2^n e_G$, where e_G is the number of induced Eulerian subgraphs of G.*

3.7 Generalized transition polynomial for graphs

For a finite set V, we define $\mathcal{P}_3(V)$ to be the set of triples (V_1, V_2, V_3) where the V_i's are pairwise disjoint and $V_1 \cup V_2 \cup V_3 = V$. Hence, (V_1, V_2, V_3) is an "ordered partition" of V (where V_i's are allowed to be empty).

Similarly as done in [26, 43], we may define a common generalization of the interlace polynomial and the global interlace polynomial.

Definition 3.17. Let G be a graph and $W = (\vec{a}, \vec{b}, \vec{c})$ with $\vec{a}$, $\vec{b}$, and $\vec{c}$ vectors indexed by $V(G)$. Then the *generalized transition polynomial* of G with respect to W is

$$Q(G, W; y) = \sum_{(X_1, X_2, X_3) \in \mathcal{P}_3(V(G))} a_{X_1} b_{X_2} c_{X_3} y^{n(A(G[X_2 \cup X_3] + X_3))},$$

where $\vec{a}$ ($\vec{b}$, $\vec{c}$, resp.) has entries a_v (b_v, c_v, resp.) for all $v \in V(G)$, $a_{X_1} = \prod_{v \in X_1} a_v$, and similarly for b_{X_2} and c_{X_3}.

Up to a simple shift of the variable y the interlace polynomial $q(G; y)$ and the polynomial $Q(G; y)$ are both specializations of $Q(G, W; y)$. Indeed, $q(G; y)$ corresponds to the case where $a_v = b_v = 1$ and $c_v = 0$ for all $v \in V(G)$, and $Q(G; y)$ corresponds to the case where $a_v = b_v = c_v = 1$ for all $v \in V(G)$.

As before, via Theorem 3.1, circle graphs form the link between the weighted transition polynomial of 4-regular graphs and the generalized transition polynomial. The ordered partition (X_1, X_2, X_3) serves as a description of the transition system (relative to an Eulerian system).

The *two-variable interlace polynomial* [7] is defined by $q(G; x, y) = \sum_{X \subseteq V} (x-1)^{r(A(G)[X])}(y-1)^{n(A(G)[X])}$, which can be obtained from $Q(G, W; y)$ by a change of variables and setting $a_v = 1, b_v = x, c_v = 0$ for all $v \in V(G)$.

4 Interlace and Tutte polynomial

For a bipartite graph G with U and V partite sets of G, we call the triple $(U, V, E(G))$ a (U, V)-*bipartite graph.*

We turn to (binary) matroids. Let B be a basis of a binary matroid M. Then the *fundamental graph* G of M with respect to B is the $(B, E(M) \setminus B)$-bipartite graph with $\{v, w\} \in E(G)$ if and only if $v \in B$ and $w \in E(M) \setminus B$ and $B \setminus \{v\} \cup \{w\}$ is a basis of M.

The next result is a generalization of Theorem 2.7.

Theorem 4.1 ([3])**.** *Let M be a binary matroid and G be the fundamental graph of M with respect to some basis B. Then $T(M; y, y) = q(G; y)$.*

In view of Theorem 4.1, Theorem 3.11, which holds for arbitrary graphs (not only bipartite graphs), provides a generalization of the evaluations of $T(M; y, y)$ for binary matroids M and $y \in \{-1, \dots, 3\}$. Indeed, it can be shown that the dimension of the bicycle space of M is equal to $n(A(G)+I)$ where G is an arbitrary fundamental graph of M — this recovers the evaluation of $T(M; -1, -1)$ from [41].

As observed by Bouchet [15], it is possible to extend Theorem 4.1 to the full two-variable Tutte polynomial $T(M; x, y)$ for binary matroids if one is careful to distinguish the two partite sets of the (bipartite) fundamental graph G of M.

We first define a graph polynomial much like the interlace polynomial, but it is only defined for (U, V)-bipartite graphs.

Theorem 4.2 ([15])**.** *Let G be a (U, V)-bipartite graph. There is a graph polynomial $q'(G; x, y)$ defined by the following relations.*

- *If $V(G) = \emptyset$, then $q'(G; x, y) = 1$.*
- *If $v \in U$ is isolated in G, then $q'(G; x, y) = x\, q'(G \setminus v; x, y)$.*
- *If $v \in V$ is isolated in G, then $q'(G; x, y) = y\, q'(G \setminus v; x, y)$.*
- *If $v \in e \in E(G)$, then $q'(G; x, y) = q'(G \setminus v; x, y) + q'((G * e) \setminus v; x, y)$.*

The next result generalizes Theorem 4.1.

Theorem 4.3 ([15])**.** *Let M be a binary matroid. Then $T(M; x, y) = q'(G; x, y)$, where G is the fundamental graph of M with respect to some basis B.*

5 Isotropic Systems and the Tutte-Martin Polynomial

The notion of isotropic system is defined by Bouchet [10] to unify various properties of transition systems of 4-regular graphs and binary matroids. In [13] two polynomials for isotropic systems are studied: the restricted Tutte-Martin polynomial and the global Tutte-Martin polynomial. It is observed in [15] that the interlace polynomial can be formulated as a specialization of the Tutte-Martin polynomial. Many results of the previous sections of this chapter can be straightforwardly obtained from the results of [13] once the interlace polynomial is formulated as a specialization of the Tutte-Martin polynomial.

We define isotropic systems in a slightly nonstandard way, using terminology from multimatroids [14] similar as done in [45]. Let U be a finite set and Ω be a partition of U such that $|\omega| = 3$ for all $\omega \in \Omega$. A *subtransversal* (*transversal*, resp.) of Ω is a subset $S \subseteq U$ such that $|S \cap \omega| \leq 1$ ($|S \cap \omega| = 1$, resp.) for all $\omega \in \Omega$. Let $\mathcal{S}(\Omega)$ and $\mathcal{T}(\Omega)$ be the sets of all subtransversals and transversals, respectively. We regard $\mathcal{S}(\Omega)$ as a vector space over $GF(2)$ isomorphic to $(GF(2)^2)^\Omega$: every singleton $\{x\} \subseteq \omega \in \Omega$ corresponds to a unique $x' \in (GF(2)^2)^\Omega$ with entries $x'_{\omega'} = (0,0)$ if $\omega' \neq \omega$ and entry $x'_\omega \neq (0,0)$. The elements of U generate in this way the whole of $\mathcal{S}(\Omega)$. We equip $\mathcal{S}(\Omega)$ with a bilinear form $B : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \to GF(2)$: for $S_1, S_2 \in \mathcal{S}(\Omega)$ we have $B(S_1, S_2) = 1$ if and only if there are an odd number of $\omega \in \Omega$ with $|\omega \cap (S_1 \cup S_2)| = 2$.

A subspace $\mathcal{L}$ of $\mathcal{S}(\Omega)$ is called *totally isotropic* if all vectors of $\mathcal{L}$ are mutually orthogonal, i.e., $B(S_1, S_2) = 0$ for all $S_1, S_2 \in \mathcal{L}$.

Definition 5.1 ([10])**.** Let Ω be as above. Then $\mathcal{I} = (\Omega, \mathcal{L})$ is an *isotropic system* if $\mathcal{L}$ is totally isotropic subspace of $\mathcal{S}(\Omega)$ of dimension $|\Omega|$.

For an isotropic system $\mathcal{I} = (\Omega, \mathcal{L})$ and $T \in \mathcal{T}(\Omega)$, we define the *nullity* of T in $\mathcal{I}$, denoted by $n_\mathcal{I}(T)$, as $\dim(\{X \in \mathcal{L} \mid X \subseteq T\})$.

Definition 5.2 ([13])**.** Let $\mathcal{I} = (\Omega, \mathcal{L})$ be an isotropic system.

- The *restricted Tutte-Martin polynomial* of $\mathcal{I}$ with respect to $T \in \mathcal{T}(\Omega)$ is $m(\mathcal{I}, T; y) = \sum_{X \in \mathcal{T}(\Omega_V), X \cap T = \emptyset} (y-1)^{n_\mathcal{I}(X)}$.
- The *global Tutte-Martin polynomial* of $\mathcal{I}$ is $M(\mathcal{I}; y) = \sum_{X \in \mathcal{T}(\Omega)} (y-2)^{n_\mathcal{I}(X)}$.

Of course, it is possible to, just as in Definition 3.17, define the obvious weighted variants of the polynomials of Definition 5.2. Let us denote the weighted variant of the global Tutte-Martin polynomial $M(\mathcal{I}; y)$ by $M(\mathcal{I}, W; y)$.

We now associate an isotropic system with a graph. Let us identify the elements of $\ker(E)$ for a $X \times Y$-matrix E over $GF(2)$ with subsets of Y in the usual way. In other words, we identify $\ker(E)$ with the cycle space of the column matroid of E. Let us denote, for $Y' \subseteq Y$, $n(E[X, Y'])$ by $n_E(Y')$.

Theorem 5.3 ([45]). *Let G be a graph, let Ω be a partition of a set with $|\omega| = 3$ for all $\omega \in \Omega$ and $|\Omega| = |V(G)|$, and let $V_1, V_2, V_3 \in \mathcal{T}(\Omega)$ be mutually disjoint. Consider the matrix*

$$E = \begin{array}{c} \begin{matrix} V_1 & V_2 & V_3 \end{matrix} \\ \begin{pmatrix} I & A(G) & A(G)+I \end{pmatrix} \end{array}.$$

Then $\mathcal{I}_G = (\Omega, \mathcal{L}_G)$ with $\mathcal{L}_G = \mathcal{S}(\Omega) \cap \ker(E)$ is an isotropic system.

Moreover, for all $X \in \mathcal{T}(\Omega)$,

$$n_{\mathcal{I}_G}(X) = n_E(X) = n(A(G + X_3[X_2 \cup X_3])),$$

where $X_i = X \cap V_i$ for all $i \in \{1, 2, 3\}$.

The graph G is called the *fundamental graph* or *graphic presentation* of $\mathcal{I}_G$ with respect to (V_1, V_2, V_3) [11]. It turns out that for every isotropic system has a fundamental graph. In fact, isotropic systems can essentially be viewed as an alternative formulation of the null spaces $\ker(E)$ (or, equivalently, the column matroids) of the matrices E as in Theorem 5.3, see [45].

As a consequence of Theorem 5.3 we have the following corollary which establishes the close relationship between the Tutte-Martin polynomials and the (global) interlace polynomial.

Corollary 5.4. *Let G be a graph and $\mathcal{I}_G$ the isotropic system from Theorem 5.3. Then we have the following:*

- $m(\mathcal{I}_G, V_3; y) = q(G; y)$, *and*
- $M(\mathcal{I}_G; y) = Q(G; y)$.

Isotropic systems have three kinds of minors which correspond to the "components" in the recursive relation of the global interlace polynomial of Theorem 3.15.

6 The interlace polynomial for arbitrary square matrices

The definition of interlace polynomial (Definition 3.3) can be straightforwardly generalized to arbitrary square matrices as follows.

Definition 6.1. Let A be a $V \times V$-matrix over some field. Then the *interlace polynomial* of A is

$$q(A; y) = \sum_{X \subseteq V} (y - 1)^{n(A[X])}.$$

With this definition in place, for a graph G, we have $q(G; y) = q(A(G); y)$, cf. Definition 3.3.

It turns out that the invariance result (Theorem 3.10) and the recursive relation (Theorem 3.8) of the interlace polynomial can be generalized to this general setting [20].

It was observed by Geelen [31] that the two operations of local complementation and edge local complementation can be seen as special cases of *principal pivot transform* (or *PPT* for short) [47]. PPT is defined with respect to any $V \times V$-matrix A over some field $\mathbb{F}$ and any $X \subseteq V$ with the principal submatrix $A[X]$ nonsingular:

$$\text{if } A = \begin{array}{cc} & \begin{array}{cc} X & V\backslash X \end{array} \\ \begin{array}{c} X \\ V\backslash X \end{array} & \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \end{array}, \text{ then } A * X = \begin{array}{cc} & \begin{array}{cc} X & V\backslash X \end{array} \\ \begin{array}{c} X \\ V\backslash X \end{array} & \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix} \end{array}.$$

The matrix $A * X[V \setminus X] = S - RP^{-1}Q$ is well known and called the *Schur complement* of $A[X]$ in A [49]. PPT is characterized by the following relation, which shows that PPT can be seen as an operation that inverts A along the entries indexed by X, i.e., PPT partially inverts A [46]:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ if and only if } A * X \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}. \tag{6.1}$$

Let G be a graph. If $v \in V(G)$ is a looped vertex, then the local complement $G*v$ of G at v has adjacency matrix $A(G)*\{v\}$. Also, if $e = \{v, w\} \in E(G)$ with v and w unlooped vertices, then the edge local complement $G * e$ of G at e has adjacency matrix $A(G) * e$. Conversely, if $A(G) * X$ is defined, then $A(G) * X$ is the adjacency matrix of a graph G' that can be obtained from G by a (possibly empty) sequence of local complement and local edge complement operations which together use each of the elements of X exactly once. In fact, any such sequence of operations defined for G results in the same graph G'. Consequently, PPT forms a common generalization of (sequences of) local complement and edge local complement. For convenience, we write $G * X$ to denote the graph with adjacency matrix $A(G * X) = A(G) * X$ (which is defined when $A(G)[X]$ is nonsingular).

For $v \in V$, we write $A \setminus v$ to denote $A[V \setminus \{v\}]$.

Theorem 6.2 ([20])**.** *Let A be a $V \times V$-matrix and $X \subseteq V$ with $A[X]$ nonsingular. Then $q(A * X; y) = q(A; y)$ and $q(A; y) = q(A \setminus v; y) + q(A * X \setminus v; y)$ for all $v \in X$.*

A special case of Theorem 6.2, where A is skew-symmetric with only zero diagonal entries and $|X| = 2$, was obtained in [32].

7 Polynomials for Delta-Matroids

We define interlace polynomials (and later weighted transition polynomials) for set systems, which include Δ-matroids and in particular matroids (here defined by their bases). Some results, in particular the recursive formulation for the global interlace polynomial (Theorem 7.14) and some evaluations (Theorem 7.15) only hold for Δ-matroids that have the additional property of being vf-safe (see Definition 7.10). The vf-safe Δ-matroids are essentially equivalent to the so-called tight 3-matroids proposed by Bouchet, see [14, 22]. Also, the class of isotropic systems (see Section 5) can be viewed as a subclass of the class of tight 3-matroids.

7.1 Delta-matroids

A *set system* (with ground set V) is a tuple $M = (V, D)$ with $D \subseteq 2^V$ a family of subsets. For simplicity we write $X \in M$ to denote $X \in D$. Also, we often simply write V to denote the ground set of the set system M under consideration. We say that M is *empty* if $D = \emptyset$. A set system is called *equicardinal* if $|X| = |Y|$ for all $X, Y \in M$. We denote by $d_M = \min_{Y \in M}(|Y|)$ the cardinality of the smallest set in M.

Let $X \subseteq V$. The *twist* (or *pivot*) of M on X, denoted $M * X$, equals $(V, D * X)$ where $D * X = \{Y \,\Delta\, X \mid Y \in M\}$.

The deletion of M by X, denoted $M \setminus X$, equals (V, D') where $D' = \{Y \in D \mid Y \cap X = \emptyset\}$. If $X = \{u\}$ is a singleton, we also write $M * u$ and $M \setminus u$ for $M * X$ and $M \setminus X$, respectively.

Notice that for a matroid M described by its bases (i.e., $M = (V, \mathcal{B})$ where $\mathcal{B}$ is the family of bases of M), we have that d_M is equal to the rank $r(M)$ of M and $M * V$ is equal to the dual matroid M^* of M.

A Δ-*matroid* [12] is a nonempty set system that satisfies the following symmetric-difference variant of the basis exchange axiom for matroids:

Definition 7.1. A nonempty set system M is a Δ-*matroid* if and only if, for each $X, Y \in M$ and $u \in X \,\Delta\, Y$, there is an element $v \in X \,\Delta\, Y$ (we allow $u = v$) such that $X \,\Delta \{u, v\} \in M$.

Let M be a Δ-matroid over V and $X \subseteq V$. Then $M * X$ is a Δ-matroid and if $M \setminus X$ is nonempty, then $M \setminus X$ is a Δ-matroid.

A set system is a matroid (described by its bases) if and only if it is an equicardinal Δ-matroid [12].

7.2 Representable delta-matroids and graphs

For a $V \times V$-matrix A (over a field $\mathbb{F}$) define the set system $\mathcal{M}_A = (V, D_A)$ with $D_A = \{X \subseteq V \mid \det A[X] \neq 0\}$. By convention $\det A[\emptyset] = 1$. Matrix A is called *skew-symmetric* if $A^T = -A$ (i.e., $a_{i,j} = -a_{j,i}$ for all $i, j \in V$). Note that we allow nonzero diagonal elements for skew-symmetric matrices over fields with characteristic 2. For a skew-symmetric matrix A, the set system $\mathcal{M}_A$ is a Δ-matroid [12].

A Δ-matroid M over V is *representable over* $\mathbb{F}$ if $M = \mathcal{M}_A * X$ for a $V \times V$-skew-symmetric A over $\mathbb{F}$ with $X \subseteq V$. A matroid turns out to be $\mathbb{F}$-representable (in this Δ-matroid sense) if and only if it is $\mathbb{F}$-representable in the usual matroid sense (as a column matroid) [12].

A (Δ-)matroid is *binary* if representable over $GF(2)$. Note that a graph G is uniquely determined by its $V(G) \times V(G)$ adjacency matrix $A(G)$. Since $A(G)$

is a (skew-)symmetric matrix over $GF(2)$, $\mathcal{M}_{A(G)}$ is a binary Δ-matroid. We write $\mathcal{M}_G$ to denote $\mathcal{M}_{A(G)}$.

Example 7.2. Let G be the graph from Example 3.5, see Figure 8. The Δ-matroid $\mathcal{M}_{A(G)}$ consists of the subsets that induce nullity 0: $\mathcal{M}_G = (\{a,b,c,d\}, \{\emptyset, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, \{a,b,c,d\}\})$.

The following theorem ties graphs and PPT to delta-matroids and twist. For notational convenience we write $G * X$ to denote $A(G) * X$. Note that $G * X$ is defined if $A(G)[X]$ is nonsingular.

Theorem 7.3. *Let G be a graph and $X \subseteq V$. Then the binary Δ-matroid $\mathcal{M}_G$ uniquely determines G (and the other way around). Moreover, $\mathcal{M}_{G*X} = \mathcal{M}_G * X$ (if the left-hand side is defined), and $d_{\mathcal{M}_G * X} = n(G[X])$.*

The statements of Theorem 7.3 are proved in [16], [12], and [21], respectively.

7.3 Interlace polynomial

We now define the interlace polynomial for set systems (and, in particular, Δ-matroids) [22].

Definition 7.4. Let M be a set system over V. The *interlace polynomial* for M is defined as

$$q(M; y) = \sum_{X \subseteq V} (y-1)^{d_{M*X}}.$$

By Theorem 7.3 we have $q(G; y) = q(\mathcal{M}_G; y)$, so the interlace polynomial for set systems generalizes the interlace polynomial for graphs. Moreover, it is easy to see that $q(M * X; y) = q(M; y)$ for all $X \subseteq V$.

Originally obtained for 4-regular graphs [34] and for binary matroids [2] we now state the connection of a weighted interlace polynomial to the Tutte polynomial for matroids in general.

Theorem 7.5 ([22])**.** *Let M be a matroid over V (described by its bases). For any values a, b, $\sum_{X \subseteq V} a^{|V \setminus X|} b^{|X|} y^{d_{M*X}} = a^{n(M)} b^{r(M)} T(M; 1 + \frac{a}{b} y, 1 + \frac{b}{a} y)$. In particular $q(M; y) = T(M; y, y)$.*

An element v of the ground set of set system M is a *loop* in M if $M * v \setminus v$ is empty and a *coloop* in M if $M \setminus v$ is empty. Note that this straightforwardly generalizes the corresponding notions of loop and coloop for matroids M. Moreover, v is said to be *nonsingular* if v is neither a loop nor a coloop in M.

Although the interlace polynomial is defined for arbitrary set systems, we only obtain recursive formulations and evaluations if we restrict to Δ-matroids.

Theorem 7.6 ([22])**.** *Let M be a Δ-matroid. If $u \in V$ is nonsingular in M, then*

$$q(M; y) = q(M \setminus u; y) + q(M * u \setminus u; y).$$

If every $v \in V$ is singular in M, then $q(M; y) = y^n$ with $n = |V|$.

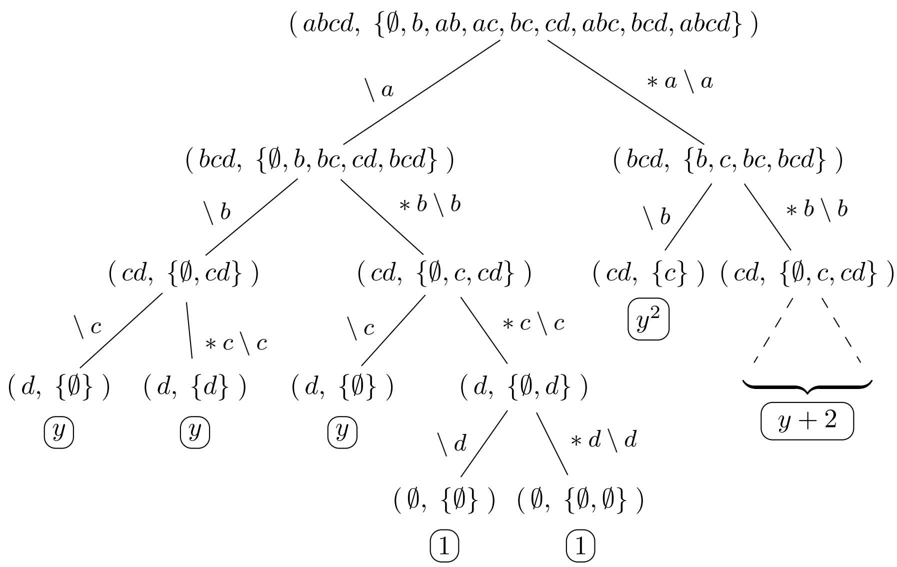

Figure 12: Recursive computation of $q(M)$; the tree rooted $(\, cd,\ \{\emptyset, c, cd\}\ )$ occurs twice.

Example 7.7. Consider the Δ-matroid $M = \mathcal{M}_G$ from Example 7.2 for the graph in Figure 8. We recursively compute its interlace polynomial $q(M; y) = y^2 + 4y + 4$, see Figure 12. Thus is the same result as in Example 3.5.

7.4 Loop complementation

The *loop complementation* of M on X, denoted $M + X$, equals (V, D') where $Y \in D'$ if and only if $|\{Z \in M \mid Y \setminus X \subseteq Z \subseteq Y\}|$ is odd [19]. The name of the latter operation is motivated by its close connection to toggling loops in graphs.

Theorem 7.8. *Let G be a graph and $X \subseteq V(G)$. Then $\mathcal{M}_{G+X} = \mathcal{M}_G + X$.*

Again, if $X = \{u\}$ is a singleton, then we also write $M + u$ for $M + X$. We assume left associativity of set system operations.

Example 7.9. Consider the Δ-matroid $M = \mathcal{M}_{A(G)}$ where G is the graph from Figure 8, see Example 7.2. Then $M + c = (\{a, b, c, d\}, \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.

For $u \in V$, the operations $*u$ and $+u$ are involutions and generate the symmetric group S_3 of permutations on three elements [19]. Moreover, twist and loop complementation commute on different elements. In this way, twist and loop complementation generate a group isomorphic to S_3^V that acts on set systems over V.

The *dual pivot* on X, denoted by $\bar{*}X$, equals $\bar{*}X = *X + X * X = +X * X + X$. More explicitly, $M \bar{*} X$ is the set system (V, D') where $Y \in D'$ if and only if $|\{Z \in M \mid Y \subseteq Z \subseteq Y \cup X\}|$ is odd.

It turns out that the class of Δ-matroids is not closed under loop complementation or dual pivot.

Definition 7.10. We say that a Δ-matroid M is *vf-safe* if applying any sequence of twist, dual pivot, loop complementation to M obtains a Δ-matroid.

It is shown that binary Δ-matroids are vf-safe [21]. Moreover, it is shown in [18] that quaternary matroids (i.e., matroids representable over $GF(4)$) are vf-safe.

7.5 Transition polynomial and global interlace polynomial

Definition 7.11. Let M be a set system. The (weighted) *transition polynomial* for M is defined as

$$Q(M, W; y) = \sum_{(A,B,C) \in \mathcal{P}_3(V)} a_A b_B c_C \, y^{d_{M * B \bar{*} C}},$$

where $W = (\vec{a}, \vec{b}, \vec{c})$ and $\vec{a}$ ($\vec{b}$, $\vec{c}$, resp.) is a vector indexed by V with entries a_v (b_v, c_v, resp.) for all $v \in V$, $a_A = \prod_{v \in A} a_v$, and similarly for b_B and c_C.

Note that the transition polynomial $Q(M, W; y)$ with weights $a_u = a, b_u = b, c_u = 0$ for each $u \in V$ is equal to the left-hand side of the equation in Theorem 7.5.

The following result shows that $Q(M, W; y)$ generalizes the generalized transition polynomial $Q(G, W; y)$ for graphs of Definition 3.17.

Theorem 7.12 ([22])**.** *Let G be a graph and W as in Definition 3.17. Then $Q(G, W; y) = Q(\mathcal{M}_G, W; y)$.*

In particular, if we define the *global interlace polynomial* $Q(M; y)$ for a set system M as $Q(M, W_1; y - 2)$ where all weights in W_1 are equal to 1, then $Q(G; y) = Q(\mathcal{M}_G; y)$. The interlace polynomial $q(M; y)$ for set systems M is equal to $Q(M, W_1; y - 1)$ with weights $a_v = 1$, $b_v = 1$ and $c_v = 0$ for all v.

The global interlace polynomial has the following very strong invariance property.

Theorem 7.13. *Let M be a set system over ground set V, and let M' be any set system obtained from M by applying a sequence of twist, loop complement and dual pivot operations. Then $Q(M'; y) = Q(M; y)$.*

We have a generic recursive relation for $Q(M, W; y)$ provided that M is a vf-safe Δ-matroid. For $v \in M$ we say that v is *strongly nonsingular* in M if both v is nonsingular in M and $M \bar{*} v \setminus v$ is nonempty.

Theorem 7.14 ([22])**.** *Let M be a vf-safe Δ-matroid and let $v \in V$.*

1. *If v is strongly nonsingular in M, then*

$$Q(M,W;y) = a_v\,Q(M\backslash v,W;y)+b_v\,Q(M*v\backslash v,W;y)+c_v\,Q(M\,\bar{*}\,v\backslash v,W;y).$$

2. *Assume v is not strongly nonsingular in M, and let $\{\ (z_1,M_1),(z_2,M_2),(z_3,M_3)\ \} = \{\ (a_v, M\setminus v),(b_v, M*v\setminus v),(c_v, M\,\bar{*}\,v\setminus v)\ \}$. If M_1 is empty, then $M_2 = M_3$ is nonempty and*

$$Q(M,W;y) = (z_2 + z_3 + z_1 y)\,Q(M_2,W;y).$$

Theorem 7.14 illustrates that vf-safe Δ-matroids have three natural types of minors, in addition to deletion (equal to $M \setminus v$ if M is not a coloop) and contraction (equal to $M * v \setminus v$ if M is not a loop) there is a third minor $M\,\bar{*}\,v \setminus v$.

Similar as for 4-regular graphs, the Penrose polynomial can also be defined in this general setting as a special case of the transition polynomial [2, 18].

7.6 Evaluations

Using Theorem 7.5, some of the evaluations of points on the diagonal of the Tutte polynomial can be generalized to set systems. We have trivially $q(M;2) = 2^{|V|}$. Moreover, $Q(M;3) = \sum_{X,Y\subseteq V, X\cap Y=\emptyset} 1 = 3^{|V|}$. Also, as $X \in M$ if and only if $d_{M*X} = 0$, we have that $q(M;1)$ is equal to the number of sets in M.

A set system is called *even* if all its sets have the same parity. Obviously a matroid (described by its bases) is even. Also, the Δ-matroid $\mathcal{M}_G$ of a graph G without loops is even.

Theorem 7.15 ([22, 13])**.** *Let M be a Δ-matroid.*

1. *If M is even and $|V| > 0$, then $q(M;0) = 0$.*
2. *If M is vf-safe, then $q(M;-1) = (-1)^{|V|}(-2)^{d_{M\,\bar{*}\,V}}$.*
3. *If M is vf-safe with $|V| > 0$, then $Q(M;0) = 0$.*
4. *If M is binary, then $q(M)(3) = k\,|q(M)(-1)|$ for some odd integer k.*

In [18] it is shown that for quaternary matroids M, $d_{M\,\bar{*}\,V}$ is equal to the dimension bd_M of the bicycle space of any representation of M, and thus we retrieve, using Theorems 7.5 and 7.15, the result from [48] that $T(M;-1,-1) = (-1)^{|E(M)|}(-2)^{bd_M}$. The case where M is binary was already shown in [41].

Example 7.16. The uniform matroid $U_{2,5}$ is not binary, but it is quaternary (i.e., representable over $GF(4)$) and therefore vf-safe. It is easy to verify that, for a subset X of the ground set V, $d_{U_{2,5}*X} = ||X|-2|$. Hence $q(U_{2,5};y) = (y-1)^3+(5+1)(y-1)^2+(10+5)(y-1)+10 = y^3+3y^2+6y$. By definition an element X is in $M\,\bar{*}\,V$ if and only if it is contained in an odd number of bases of M. Straightforward combinatoric arguments show that $U_{2,5}\,\bar{*}\,V = U_{2,5}$. Indeed $q(U_{2,5};-1) = -4 = (-1)^5(-2)^2$, cf. Theorem 7.15.

section	structure \ # of directions	2	3	3 weighted
2	2-in 2-out or 4-regular graph $\vec{G}$, G	$m(\vec{G};y)$	$M(G;y)$	$M(G,W;y)$
3	graph G	$q(G;y)$	$Q(G;y)$	$Q(G,W;y)$
4, 7	matroid M	$T(M;y,y)$	-	-
5	isotropic system $\mathcal{I}$	$m(\mathcal{I},T;y)$	$M(\mathcal{I};y)$	$M(\mathcal{I},W;y)$
6	square matrix A	$q(A;y)$	-	-
7	(vf-safe) Δ-matroid M	$q(M;y)$	$Q(M;y)$	$Q(M,W;y)$

Table 1: Summary of the main polynomials considered in this chapter.

8 Discussion and Open Problems

Table 1 lists the main polynomials considered in this chapter. The three right-most columns arrange the polynomials according to the number of "directions" or minors in its recursive formulas. Notice that, apart from the rows on $T(M;y,y)$ for matroids M and $q(A,y)$ for square matrices A, the rows are ordered in increasing level of generality. Moreover, the last column, concerning 3 directions with weights, generalizes the other two columns.

We mention that the notion of multimatroid, introduced by Bouchet [14], generalizes the notion of Δ-matroid by allowing an arbitrary number of directions (instead of two, and, in the case of vf-safe Δ-matroids, three). It turns out that the transition polynomial $Q(M,W;y)$ (and thus also its specializations) can be defined for arbitrary multimatroids. Moreover, many properties of $Q(M,W;y)$, such as its recursive formulation and some evaluations, carry over to this multimatroid polynomial [8, 22].

Many open problems and research directions remain. For example, in [28, 9] the Martin polynomial $m(G;y)$ is considered for arbitrary Eulerian (di)graphs G. A natural question is to generalize this polynomial similar as done in this chapter for the cases of 2-in 2-out and 4-regular graphs. A difficulty here is that a vertex reduction may split a connected graph in more than two connected components. This increases the "nullity" by more than one, which is impossible for the standard elementary minors of contraction and deletion in (delta-)matroids. Perhaps the generalization of the notion of delta-matroid called *parity system* defined in [17] provides a clue for a suitable generalization to incorporate graphs more general than 2-in 2-out and 4-regular graphs.

Another direction for further research is to generalize the full two-variable Tutte polynomial to polynomials defined on more general structures than matroids. While part of the (x,y)-plane is generalized through the interlace polynomial of Δ-matroids through Theorem 7.5, it is an open question to generalize the whole (x,y)-plane (i.e., the whole two-variable Tutte polynomial). Perhaps Theorem 4.2 provides a lead to generalize the whole Tutte polynomial to more general structures than matroids.

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